where

$$\langle f_p, g_{np}, w_{p0}, w_{p1} \rangle = \frac{1}{N_p} \sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} \left\langle (1/\rho^{(l)})F^{(l)}, \sum_{m=1}^{4} C_{mnl}\, g_{ml}, w^{(l)}(0), w^{(l)}(0)_{,t} \right\rangle \rho^{(l)} \xi_p^{(l)}\, dx \quad (25)$$

such that

$$N_p = \sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} \rho^{(l)} \xi_p^{(l)} \xi_p^{(l)}\, dx \quad (26)$$

Substituting Eqs. (10, 21, and 22) into Eq. (7) gives

$$\sum_{p=1}^{\infty} \left\{ [(EI)^{(l)} \xi_{p,xx}^{(l)} \zeta_p]_{,xx} - \xi_p^{(l)} f_p + \rho^{(l)} \xi_p^{(l)} \zeta_{p,tt} + \rho^{(l)} \sum_{n=1}^{2L+2} g_{np} h_{n,tt}\, \xi_p^{(l)} \right\} = 0; \; l = 1, 2, \ldots, L \quad (27)$$

Now applying the piecewise-weighted orthogonality principle, Eq. (20), yields

$$\sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} \text{Eqs. (27)} \quad \xi_p^{(l)}\, dx \to \zeta_{p,tt} + \Omega_p^2 \zeta_p = f_p - \sum_{n=1}^{2L+2} g_{np} h_{n,tt} \quad (28)$$

In terms of the reduced initial conditions, Eq. (25), the solution of Eq. (28) takes the form

$$\zeta_{p(t)} = w_{p0} \cos \Omega_p t + \frac{w_{p1}}{\Omega_p} \sin \Omega_p t + \frac{1}{\Omega_p} \int_0^t \left[f_p(\tau) - \sum_{n=1}^{2L+2} g_{np} h_n(\tau)_{,\tau\tau} \right] \sin \Omega_p (t - \tau)\, d\tau \quad (29)$$

Hence $w^{(l)}$ is finally given by

$$w^{(l)}(x, t) = \sum_{p=1}^{\infty} \xi_p^{(l)} \zeta_p + \sum_{n=1}^{2L+2} \sum_{m=1}^{4} C_{mnl}\, g_{ml}\, h_n \quad (30)$$

Discussion

Because of the form of the eigenfunction expansion given by Eq. (10), the solution of Eqs. (1–3) has been reduced basically to the problem of obtaining the eigenvalues of Eqs. (13–15). To simplify the search for the eigenvalues of these equations, the realness and positive definiteness of Ω^2 will be established.

To prove the realness of Ω^2 assume that

$$\begin{aligned} \xi^{(l)} &= Re\{\xi^{(l)}\} + jIm\{\xi^{(l)}\} \\ \Omega^2 &= Re\{\Omega^2\} + jIm\{\Omega^2\} \end{aligned} \quad (31)$$

where $j = (-1)^{1/2}$. In the spirit of the development of the piecewise-weighted orthogonality procedure, Eq. (20), it can be shown that in terms of Eqs. (31), Eq. (13) can be directly manipulated to yield

$$\begin{aligned} &Im\{\Omega^2\} \sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} \rho^{(l)} (Re^2\{\xi^{(l)}\} + Im^2\{\xi^{(l)}\})\, dx = \\ &\sum_{l=1}^{L} [Re\{\xi^{(l)}\}((EI)^{(l)} Im\{\xi^{(l)}\}_{,xx})_{,x} - \\ &Im\{\xi^{(l)}\}((EI)^{(l)} Re\{\xi^{(l)}\}_{,xx})_{,x} + \\ &Im\{\xi^{(l)}\}_{,x} (EI)^{(l)} Re\{\xi^{(l)}\}_{,xx} - \\ &Re\{\xi^{(l)}\}_{,x} (EI)^{(l)} Im\{\xi^{(l)}\}_{,xx}] \Big|_{x_l}^{x_{l+1}} \end{aligned} \quad (32)$$

Furthermore, using Eqs. (31) in conjunction with the interbeam conditions, Eqs. (14), the following identities can be established:

$$\begin{aligned} &Re\{\xi^{(l)}\} [(EI)^{(l)} Im\{\xi^{(l)}\}_{,xx}]_{,x} - Im\{\xi^{(l)}\} [(EI)^{(l)} Re\{\xi^{(l)}\}_{,xx}]_{,x} + \\ &Im\{\xi^{(l)}\}_{,x} (EI)^{(l)} Re\{\xi^{(l)}\}_{,xx} - Re\{\xi^{(l)}\}_{,x} (EI)^{(l)} Im\{\xi^{(l)}\}_{,xx} = \\ &Re\{\xi^{(l+1)}\} [(EI)^{(l+1)} Im\{\xi^{(l+1)}\}_{,xx}]_{,x} - \\ &Im\{\xi^{(l+1)}\} [(EI)^{(l+1)} Re\{\xi^{(l+1)}\}_{,xx}]_{,x} + \\ &Im\{\xi^{(l+1)}\}_{,x} (EI)^{(l+1)} Re\{\xi^{(l+1)}\}_{,xx} - \\ &Re\{\xi^{(l+1)}\}_{,x} (EI)^{(l+1)} Im\{\xi^{(l+1)}\}_{,xx} \end{aligned} \quad (33)$$

Substituting Eqs. (14) and (33) into Eq. (32) yields

$$Im\{\Omega^2\} \sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} \rho^{(l)} (Re^2\{\xi^{(l)}\} + Im^2\{\xi^{(l)}\})\, dx \equiv 0 \quad (34)$$

Since all the integrals appearing in Eq. (34) are positive definite, it follows that $Im\{\Omega^2\} \equiv 0$, thus Ω^2 is purely real.

To establish that Ω^2 is positive definite, after several manipulations, the variational formulation of Eqs. (14) and (15) can be used to develop the following form of Rayleigh's quotient

$$\Omega^2 = \frac{\sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} (EI)^{(l)} \xi_{,xx}^{(l)} \xi_{,xx}^{(l)}\, dx}{\sum_{l=1}^{L} \int_{x_l}^{x_{l+1}} \rho^{(l)} \xi^{(l)} \xi^{(l)}\, dx} \quad (35)$$

Since both the numerator and denominator of Eq. (35) are positive definite, $\Omega^2 > 0$.

Because of the generality of the procedure developed herein, branched beam systems can also be handled. This is possible through the straight forward modification of the piecewise-weighted orthogonality relation depicted by Eq. (19) along with the inclusion of torsional effects. In fact the orthogonalization procedure can also be extended to Timoshenko-type theory as well as to composite plate and shell configurations with and without branches.

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Oblique Compressible Sears Function

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Introduction

THIS Note concerns the lift response of a thin, infinite-span wing flying subsonically through a stationary sinusoidal gust at an arbitrary angle to the lines of constant phase: that is, the generalization of Sears' classical result[1] to oblique gusts and compressible flow.

With the freestream U in the positive x direction and the wing of chord $2b$ along the y axis, the upwash on the wing is

$$w(x, y, t) = w_o \exp\{-i[k_1(x - Ut)/b + k_2 y/b]\}, \quad |x| \leqq b \quad (1)$$

The corresponding lift distribution is governed by three nondimensional parameters: chordwise wavenumber (or reduced frequency) k_1, spanwise wavenumber k_2, and Mach number M (w_o/U, small by hypothesis, is unimportant as the solution is linear therein). We *assume* that the usual linearized equation for the velocity potential is valid for all combinations of parameters of interest.

Sears' solution (for $k_2 = M = 0$) was extended by Osborne[2] to

Received March 22, 1974; revision received May 20, 1974.

Index categories: Nonsteady Aerodynamics; Subsonic and Transonic Flow.

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subsonic flow ($k_2 = 0$, $M \neq 0$) by a partially heuristic analysis valid for small values of the perturbation parameter $k_1 M/(1-M^2)$. Chu and Widnall recently presented a formal series expansion of the full problem (k_1, k_2, M all variable)[3] in terms of this same small parameter. Their result exactly corresponds to Osborne's for $k_2 = 0$ and controverts the accuracy of previously published numerical results[4,5] not in agreement for vanishing values of the small parameter.

The analytical form of Chu and Widnall's solution depends on whether $k_2/(1-M^2)^{1/2}$ is small or large compared to the perturbation parameter: a condition roughly corresponding to whether the component of the gust's convection speed in the direction of the constant phase lines is supersonic or subsonic. (The intermediate transonic case is covered by extending the solution from both sides and accepting a small discontinuity.)

The "subsonic" form (termed "3D" in Ref. 3) is in fact valid [compare Eq. (12) of Ref. 4] under the single, less restrictive condition

$$k_2 \gg k_1 M/(1-M^2)^{1/2}$$

that is, for large enough k_2 when M and k_1 are fixed. Values computed from this expression for small k_2 (for which there is no a priori reason to suppose it valid) are numerically close to values computed using the proper ("supersonic" or "quasi-2D") form. As a matter of fact the expression can also be obtained through straightforward extension of Osborne's heuristic analysis for $k_2 \neq 0$; setting $k_2 = 0$ recovers Osborne's original result, valid for all k_1 when M is small enough. Thus Chu and Widnall's "3D" expression is applicable (within usual linearized theory) to all wavenumbers for small M.

The remainder of this Note recasts this expression [Eq. (33) of Ref. 3] into the convenient form (claimed[6] to be particularly suited to the calculation of turbomachinery noise) of the Sears function multiplied by a number of factors. [With the appropriate substitutions and allowance for slightly differing notation, Eq. (4) below is equivalent to Eq. (33) of Ref. 3.] The physical interpretation of each factor is noted; useful algebraic approximations to the rather complex Bessel function expressions are suggested.

Aerodynamic Transfer Function

The lift coefficient (or any other linear response) can be conveniently regarded as the output due to an "aerodynamic transfer function" (a term apparently coined by B. Etkin) operating on a known input. Here the input is taken to be the angle of attack at midchord

$$\alpha(y,t) = w(0,y,t)/U \tag{2}$$

The instantaneous lift coefficient at spanwise position y is then written as

$$C_l(y,t) = T_A(k_1,k_2,M)\alpha(y,t) \tag{3}$$

where T_A is the aerodynamic transfer function (i.e., a generalized lift curve slope). The expression from Ref. 3 discussed previously may be manipulated into the form

$$T_A(k_1,k_2,M) = a_o \beta^{-1} S(k_1/\beta^2) F(k_1/\beta^2, k_2/\beta, M) \tag{4}$$

where

$$a_o = T_A(0,0,0)$$

is the two-dimensional steady lift curve slope—theoretically equal to 2π; in practice best replaced by its measured value.

The Prandtl-Glauert factor

$$\beta^{-1} = (1-M^2)^{-1/2} = T_A(0,0,M)/a_o \tag{5}$$

corrects the steady-state lift curve slope for compressibility causing it to increase with increasing Mach number. All other compressibility effects tend to decrease the lift curve slope.

The Sears function

$$S(k_1) = T_A(k_1,0,0)/a_o \tag{6}$$

gives the two-dimensional incompressible sinusoidal gust response.

The final factor F accounts for the gust's obliqueness and unsteady compressible effects. The dependence can be displayed more explicitly through the further factoring of F into

$$F(K_1,K_2,M) = F_1[(M^4K_1^2+K_2^2)^{1/2}, \tan^{-1}(K_2/M^2K_1)] \times F_2[(K_1^2+K_2^2)^{1/2}, \tan^{-1}(K_1/K_2)] \tag{7}$$

with

$$F_1(M^2k_1/\beta^2, 0) = T_A(k_1,0,M)/[a_o\beta^{-1}S(k_1/\beta^2)] \tag{8}$$

corresponding to Osborne's extension of Sears' result to subsonic flow[2] being the only factor to display the Mach number explicitly: thus F_1 may be termed the "unsteady compressibility factor." The remaining factor

$$F_2[(k_1^2+k_2^2)^{1/2}, \tan^{-1}(k_1/k_2)] = T_A(k_1,k_2,0)/[a_o S(k_1)] \tag{9}$$

accounts for the obliqueness of the sinusoidal gust in incompressible flow. Hence F_2 is an "obliqueness factor" modifying Sears function for spanwise upwash variations.

Manipulation of Chu and Widnall's "3D" expression[3] leads (taking due account of minor difference in notation) to the functional forms (shown on Figs. 1 and 2)

$$F_1(r,\theta) = [J_o(re^{i\theta}) - iJ_1(re^{i\theta})]/[I_o(r\sin\theta)+I_1(r\sin\theta)] \tag{10}$$

the J_n and I_n being Bessel functions in usual notation, and

$$F_2(k,\lambda) = T(k,\lambda)/T(k,\pi/2) \tag{11}$$

where T is the (normalized) incompressible oblique gust transfer function.[7] Note that $T(k,\pi/2) \equiv S(k)$. An expression differing from the exact linearized theory solution only through a few small amplitude excursions in the region $k \sim 0(1)$ is[7]

$$T(k,\lambda)^{-1} \approx \tfrac{1}{2}k\left[\pi - i\int_{-\infty}^{\infty}(1+\sin\lambda t+t^2)^{1/2}t^{-1}e^{ikt}\,dt\right] \times F_1^*(k,\pi/2-\lambda) \tag{12}$$

F_1^* being the complex conjugate of the function defined by Eq. (10). Reference 7 gives a series expansion and tabulates values for the integral in square brackets.

Algebraic Approximations

Computation of the Bessel function expressions [particularly with complex argument, as in Eq. (10)] can be rather lengthy—even on electronic computers. For almost all practical purposes the functional forms may be replaced by readily computable algebraic approximations—matching the original expression to within a few percent for any combination of arguments. Approxi-

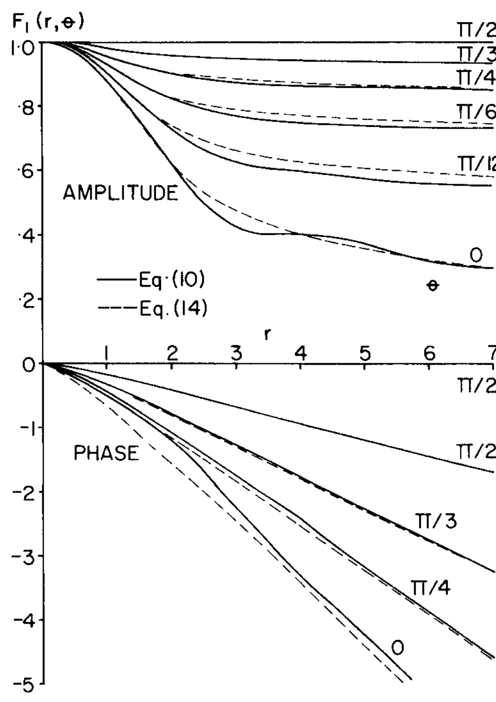

Fig. 1 Unsteady compressibility factor: $F_1(r,\theta)$.

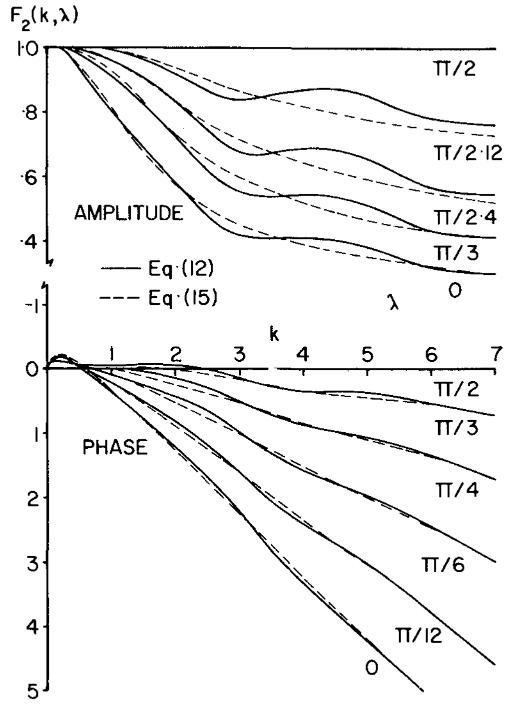

Fig. 2 Obliqueness factor: $F_2(k, \lambda)$.

mations should, of course, reduce to the correct asymptotic forms in limiting cases.

An excellent approximation to the Sears function is[8]

$$S(k) \approx [(k+0.1811)/(0.1811+1.569k+2\pi k^2)]^{1/2} \times \exp\{ik[1-(\tfrac{1}{2}\pi^2)/(1+2\pi k)]\} \qquad (13)$$

In very few applications (i.e., calculation of $\dot{\alpha}$ derivatives[9]) this expression's failure to exactly account for the Sears functions logarithmic behavior as $k \to 0$ may lead to inconsistencies.

The following asymptotically correct expressions (indicated by broken lines on Figs. 1 and 2) may be used to replace Eqs. (10) and (11)

$$F_1(r,\theta) \approx \left[\frac{4\pi+2\cos^2\theta r^3+\pi\sin\theta r^4}{4\pi+\pi\cos^2\theta r^2+\pi r^4}\right]^{1/2} \times \exp\left\{-ir\cos\theta\left[\frac{\frac{1}{2}(\pi/2-\theta)+r\cos\theta}{(\pi/2-\theta)+r\cos\theta}\right]\right\} \qquad (14)$$

$$F_2(k,\lambda) \approx \left[\frac{2\pi+r^3}{1+\pi r^2+\frac{1}{2}\pi r^4}\,\frac{2\pi+\pi\sin^2\lambda r^2+\frac{1}{2}\pi r^4}{2\pi+\sin^2\lambda r^3+\frac{1}{2}\pi\cos\lambda r^4}\right]^{1/2} \times \exp[\![ik\{1-\sin\lambda-\pi^2/(2+4\pi k)+\tfrac{1}{2}\pi\lambda(2+\cos\lambda)/[1+\pi k(2+\cos\lambda)]\}]\!] \qquad (15)$$

Equation (14) with $\theta = 0$ can be used to form a convenient algebraic approximation to Osborne's result.[2] Equation (15) incorporates a better approximation to Eq. (12) than the one suggested (mainly because of its simplicity) in Ref. 7 and used in Ref. 3. Combination of Eq. (15) with the exact expression for the Sears function gives the asymptotically correct linearized incompressible result for all extreme values of k_1 and k_2.

Conclusions

These results, with Eqs. (14) and (15), provide a formula that interpolates between the known analytical expressions for the oblique compressible Sears function. With $M = 0$ the exact linearized result is recovered for all k_1 if $k_2 \to 0$ or $k_2 \to \infty$. In subsonic flow nonzero values to k_1 and/or k_2 reduce the lift coefficient; mitigating the increase in surface velocities accompanying increasing Mach number in steady, two-dimensional flow and delaying the onset of transonic effects. Computed values indicate that the increase in lift curve slope with increasing M is reversed (that is T_A becomes less than a_o) if $M > 2k_1^{1/2}$. Correspondence with accurate numerical solutions of the usual linearized formulation is not necessarily the best basis for determining limits of applicability. Validity of the usual formulation at high frequency (large k_1) may be questioned with regard to both linearization of the potential equation[10] and application of the Kutta condition.[11] Indirect experimental verification could ensue from spectral analysis of airfoil response to grid turbulence or other unsteady inputs.

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Extension of the Momentum Integral Method to Three-Dimensional Viscous-Inviscid Interactions

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Introduction

A NUMBER of methods have been advanced for investigating two-dimensional laminar viscous-inviscid interactions. One of the most successful is the method of Lees and Reeves[1] as developed by Klineberg[2] and Georgeff.[3]

The development of methods capable of predicting the main features of three-dimensional viscous-inviscid interactions, however, has been limited. Leblanc, Horton, and Ginoux[4] developed a method based on that of Lees and Reeves for investigating axisymmetric flow with spin. However, their method was limited

Received April 29, 1974. This work was conducted at the Imperial College of Science and Technology, London.

Index categories: Boundary Layers and Convective Heat Transfer – Laminar; Jets, Wakes, and Viscid-Inviscid Flow Interactions; Supersonic and Hypersonic Flow.

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